1st Course Beijing
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Introduction:

What is a p-adic Hodge structure

Recall: * Real Hodge structure (pure) of weight w:

V = finite dim. $\mathbb{R}$-vector space

+ bigrading $V_{\mathbb{C}} = \bigoplus_{p+q=w} V_{\mathbb{C}}^{p,q}$

s.t. $\overline{V_{\mathbb{C}}^{p,q}} = V_{\mathbb{C}}^{q,p}$

* $X/\mathbb{C}$ proper smooth alg. variety

$H^i(X,\mathbb{R})$ equipped with a real Hodge structure of weight i

p-adic setting: Plenty of different structure / results

Hodge-Tate Galois representations
Crystalline " "
de Rham "
filtered φ-modules
Breuil-Kisin-modules
(φ,Γ)-modules
⋮

+ Comparison theorems for proper smooth alg. var. / $\mathbb{Q}_p$

$\rightsquigarrow$ this is a mess: not clear what is a p-adic Hodge structure

Solution: Come back to real Hodge structures

Simpson geometric point of view of twistors:

$$\widetilde{\mathbb{P}}^1_{\mathbb{R}} = \mathbb{P}^1_{\mathbb{C}} / z \sim -\frac{1}{\bar{z}}$$

twisted form of $\mathbb{P}^1_{\mathbb{R}}$ without real point

$\parallel$

Conic without real point

$\parallel$

Severi-Brauer variety attached to $\mathbb{H}$

* Equipped with a point $\infty \in \widetilde{\mathbb{P}}^1_{\mathbb{R}}$

$$\begin{array}{ccc} 0 & & \infty \\ & \searrow \swarrow & \\ & \infty & \end{array} \qquad \begin{array}{c} \mathbb{P}^1_{\mathbb{C}} \\ \downarrow \pi \\ \widetilde{\mathbb{P}}^1_{\mathbb{R}} \end{array} \Big) \mathbb{Z}/2\mathbb{Z}$$

* $\mathbb{P}^1_{\mathbb{C}} \circlearrowleft \mathbb{C}^\times$ via $z \mapsto \lambda z$ for $\lambda \in \mathbb{C}^\times$

Descends to an action of $U(1)$ on $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$

∞ = unique point with a finite orbit

$$\begin{tikzcd} \mathbb{P}^1_{\mathbb{C}} \arrow[d, "\pi"'] & \circlearrowleft \; \mathbb{C}^\times \\ \widetilde{\mathbb{P}}^1_{\mathbb{R}} & \circlearrowleft \; U(1) \arrow[u, hook] \end{tikzcd}$$

$U(1)$ stabilizes ∞

* Classification of vector bundles / $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$

$\lambda \in \frac{1}{2}\mathbb{Z}$. Define $\underbrace{\mathcal{O}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}}(\lambda)}_{}= \begin{cases} \pi_* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2\lambda) & \text{if } \lambda \notin \mathbb{Z} \\ \underbrace{\mathcal{L}}_{\text{line bundle}} \text{ s.t. } \pi^*\mathcal{L} = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2\lambda) & \text{if } \lambda \in \mathbb{Z} \end{cases}$

$\hookrightarrow$ slope λ stable vector bundle on $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$

<u>Prop:</u> $\{\lambda_1 \geq \dots \geq \lambda_n \mid \lambda_i \in \frac{1}{2}\mathbb{Z}, n \in \mathbb{N}\} \xrightarrow{\sim} \mathrm{Bun}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}}}$

$(\lambda_i)_i \longmapsto \left[\bigoplus_i \mathcal{O}(\lambda_i)\right]$

i.e. (1) Slope λ v.b. $= \bigoplus_{\text{finite}} \mathcal{O}(\lambda)$

(2) The Harder–Narasimhan filtration of a v.b. is split.

In particular: $\mathrm{Vect}_{\mathbb{R}} \xrightarrow{\sim} \{\text{slope } 0 \text{ semi-stable v.b.}\}$, with inverse $H^0(\widetilde{\mathbb{P}}^1_{\mathbb{R}}, -)$

$V \longmapsto V \otimes_{\mathbb{R}} \mathcal{O}$

$V \in \mathrm{Vect}_{\mathbb{R}}$ - Suppose $V_{\mathbb{C}}$ equipped with a filtration

$\mathrm{Fil}^{\cdot} V_{\mathbb{C}}$ (decreasing, "finite")

$\mathbb{C}[[t]]$-lattice

Define a lattice Λ in $V_{\mathbb{C}}((t)) := V_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}((t))$ by the formula

equipped with filtration

$\mathrm{Fil}^i \mathbb{C}((t)) = t^i \mathbb{C}[[t]]$

$$\Lambda = \mathrm{Fil}^0 \left(V_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{C}((t)) \right)$$

$$:= \sum_{i \in \mathbb{Z}} \mathrm{Fil}^i V_{\mathbb{C}} \otimes_{\mathbb{C}} t^{-i} \mathbb{C}[[t]]$$

t = uniformizing element of $\widehat{\mathbb{P}}^1_{\mathbb{R}}$ at ∞.

$\circlearrowleft$
$U(1)$ via $t \mapsto \lambda t$
$\lambda \in U(1)$

via Beauville-Laszlo

Then Λ defines a modification of vector bundles

$$\left(V \otimes_{\mathbb{R}} \mathcal{O}_{\widehat{\mathbb{P}}^1_{\mathbb{R}}}\right)\Big|_{\widehat{\mathbb{P}}^1_{\mathbb{R}} \setminus \infty} \xrightarrow{\sim} \mathcal{E}|_{\widehat{\mathbb{P}}^1_{\mathbb{R}} \setminus \infty}$$

with $\widehat{\mathcal{E}}_{\infty} = \Lambda$

This is moreover $U(1)$-equivariant.

This induces a bijection for $V \in \mathrm{Vect}_{\mathbb{R}}$

$$\left\{ \text{Filtrations on } V_{\mathbb{C}} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} U(1)\text{- equivariant modifications} \\ V \otimes_{\mathbb{R}} \mathcal{O}_{\mathbb{P}^1_{\mathbb{R}}} \rightsquigarrow \mathcal{E} \end{array} \right\}$$

And thus

$$\left\{ (V, \mathrm{Fil}^{\cdot} V_{\mathbb{C}}) \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} U(1)\text{-eq. modifications of } U(1)\text{-eq. v.b.} \\ \mathcal{E}_1 \rightsquigarrow \mathcal{E}_2 \text{ with } \mathcal{E}_1 \text{ s.s. slope } 0 \\ \text{and } H^0(\mathcal{E}_1) \circlearrowleft \text{ trivial action of } U(1). \end{array} \right\}$$

<u>Prop</u> (twistor interpretation of Hodge structures):
$(V, \mathrm{Fil}^{\cdot} V_{\mathbb{C}})$ defines a weight w pure Hodge structure,
i.e. $\mathrm{Fil}^{\cdot} V_{\mathbb{C}}$ and $\overline{\mathrm{Fil}^{\cdot} V_{\mathbb{C}}}$ are w-opposite,
iff in the corresponding modification $\mathcal{E}_1 \rightsquigarrow \mathcal{E}_2$
$\mathcal{E}_2$ is semi-stable slope $\frac{w}{2}$ i.e. $\simeq \mathcal{O}\left(\frac{w}{2}\right)^m$, $m \in \mathbb{N}$.

Thus:

$$\left[\begin{array}{c}\{\text{weight } w \text{ pure Hodge structures}\}\\ \wr\!\wr \\ \left\{\begin{array}{l} U(1)\text{-equivariant modifications } \mathcal{E}_1 \rightsquigarrow \mathcal{E}_2 \\ \text{s.t. } \mathcal{E}_1 \text{ s.s. slope } 0,\ \mathcal{E}_2 \text{ s.s. slope } \frac{w}{2},\ \text{and } \underset{\text{trivial } U(1)\text{-action}}{H^0(\widetilde{\mathbb{P}}^1_{\mathbb{R}}, \mathcal{E}_1) \circlearrowleft} \end{array}\right\}\end{array}\right]$$

We are going to do the same in the p-adic setting:

$$\widetilde{\mathbb{P}}^1_{\mathbb{R}} \longleftrightarrow X \text{ the curve}$$

$$\widetilde{\mathbb{P}}^1_{\mathbb{R}} \circlearrowleft U(1) \longleftrightarrow X \circlearrowleft Gal(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$$

$$\mathbb{C}[[t]] = \widehat{\mathcal{O}}_{\widetilde{\mathbb{P}}^1_{\mathbb{R}},\infty} \longleftrightarrow \widehat{\mathcal{O}}_{X,\infty} = B^+_{dR} \qquad t = \text{"Fontaine's } 2i\pi\text{"}$$

$$\lambda \in U(1)\ \ \lambda.t = \lambda t \longleftrightarrow \sigma \in Gal(\overline{\mathbb{Q}}_p|\mathbb{Q}_p) \qquad \sigma(t) = \chi_{cyc}(\sigma)\, t$$

$$\begin{array}{c}\mathbb{P}^1_{\mathbb{C}} \\ \downarrow \ \big)\mathbb{Z}/2\mathbb{Z} \\ \widetilde{\mathbb{P}}^1_{\mathbb{R}}\end{array} \longleftrightarrow \begin{array}{c} X_\infty \\ \downarrow \ \big)\widehat{\mathbb{Z}}\text{-pro-Galois cover} \\ X\end{array}$$

Classification théo. for v.b. $/\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ $\longleftrightarrow$ Classification théo. for v.b. $/X$

The curve

2 versions of the curve: X^{ad} adic version analog of p-adic Riemann surface

X schematical version analog of a proper smooth algebraic curve

+ GAGA analytification morphism $X^{ad} \longrightarrow X$
and an "ample" line bundle $\mathcal{O}(1)$ on X^{ad} such that

$$X = \mathrm{Proj}\left(\bigoplus_{d \geq 0} H^0(X^{ad}, \mathcal{O}(d))\right)$$

Both rely on the construction of an intermediate space Y = adic space
$\circlearrowleft$
φ action of a "crystalline" Frobenius

Hopefully: Do not need to know what's an adic space

Since "Y is Stein" $\Rightarrow$ Completely determined by the Fréchet algebra $\underbrace{\mathcal{O}(Y)}$

"holomorphic functions of the variable p"

Holomorphic functions of the variable p

Hypothesis: * $\mathbb{F}_q$ finite field

* E local field with residue field $\mathbb{F}_q$

$\to$ $[E : \mathbb{Q}_p] < +\infty$ "classical p-adic Hodge theory case" wrt. coefficients in E

$\to$ $E = \mathbb{F}_q((\pi))$ "Hartl-Pink case" of equal char. local Shtukas

π uniformizing element of E

(everything in the following won't depend on the choice of π but let's fix it)

$*$ $F|\mathbb{F}_q$ Complete valued field
$q^{-v(\cdot)} = |\cdot| : F \to \mathbb{R}_+$ non trivial
perfect $\Rightarrow$ v non discrete

] F = char. p perfectoid field

Ex: $* F = k((T^{1/p^\infty}))$ $k|\mathbb{F}_q$ perfect

$\widehat{\text{Completion of}}$ $\bigcup_{n \geq 1} k((T^{1/p^n}))$

$*$ $F = \widehat{k((T))}$

$*$ F spherically complete

$= k((T^\Gamma)) = \left\{ f = \sum_{\alpha \in \Gamma} a_\alpha T^\alpha \;\middle|\; \begin{array}{l} a_\alpha \in k \\ \text{supp}(f) \text{ well ordered} \end{array} \right\}$

$\Gamma \subset \mathbb{R}$ additive subgroup satisfying $p\Gamma = \Gamma$

$\rightarrow$ see Poonen "Maximally complete fields"

Rem: One could take $F = K^\flat$ with $K|E$ perfectoid field but we don't want to fix such a K since, as we will see later, the curve is "the moduli of untilts of F"

The ring A_{inf}:

$$\mathbb{A} := A_{inf} = \begin{cases} W_{O_E}(O_F) & \text{if } E|\mathbb{Q}_p \\ O_F \hat{\otimes}_{\mathbb{F}_q} O_E = O_F[[\pi]] & \text{if } E = \mathbb{F}_q((\pi)) \end{cases}$$

Here W_{O_E} = ramified Witt vectors

$= W \otimes_{O_{E_0}} O_E$ if $E_0|\mathbb{Q}_p$ is the maximal unramified extension of $\mathbb{Q}_p$ in E

O_F perfect

$$\Rightarrow \mathbb{A} = \Big\{ \overbrace{\sum_{n\geq 0}[x_n]\pi^n}^{\text{holo. function variable } \pi \text{ coefficients in } F} \;\Big|\; x_n \in O_F \Big\} \quad \text{unique writing}$$

where $[x]=x$ if $E=\mathbb{F}_q((\pi))$ i.e. in the equal char. case $[-]$ is additive

when $E|\mathbb{Q}_p$

$$\sum_{n\geq 0}[x_n]\pi^n+\sum_{n\geq 0}[y_n]\pi^n=\sum_{n\geq 0}\left[P_n(x_0,\dots,x_n,y_0,\dots,y_n)\right]\pi^n$$

$$P_n\in\mathbb{F}_q\left[X_0^{1/p^\infty},\dots,X_n^{1/p^\infty},Y_0^{1/p^\infty},\dots,Y_n^{1/p^\infty}\right]$$

Same for multiplication

<u>Ex:</u> $E=\mathbb{Q}_p$. $\quad P_0=X_0+Y_0$

$$P_1=X_1+Y_1+S\left(X_0^{1/p},Y_0^{1/p}\right)$$

$$S(X,Y)=\frac{(X+Y)^p-X^p-Y^p}{p}$$

<u>The topology on $\mathbb{A}$:</u> Fix $\varpi\in F$ s.t. $0<|\varpi|<1$

$\mathbb{A}$ is equipped with the $(\pi,[\varpi])$-adic topology (complete for this topology)

via
$$\mathcal{O}_F^{\mathbb{N}}\xrightarrow{\sim}\mathbb{A}$$
$$(x_n)_{n\geq 0}\longmapsto\sum_{n\geq 0}[x_n]\pi^n$$

This is the product topology on $\mathcal{O}_F^{\mathbb{N}}$ i.e. the topology of weak convergence of Teichmüller coefficients.

<u>Then:</u> $$Y = \mathrm{Spa}(\mathbb{A}, \mathbb{A}) \setminus V(\pi[\varpi])$$
$$\big\downarrow \text{open}$$
$$\mathcal{Y} = \mathrm{Spa}(\mathbb{A}, \mathbb{A}) \setminus V(\pi, [\varpi])$$
$$= \mathrm{Spa}(\mathbb{A}, \mathbb{A})_a \quad \text{(analytic points)}$$

$\mathcal{Y}$ = "compactification of Y"

$$Y = \mathcal{Y} \setminus \left(V(\pi) \cup V([\varpi]) \right)$$ remove 2 boundary divisors.

<u>What is $\mathcal{O}(Y)$?</u>

Start with $\mathbb{A}[\frac{1}{\pi}, \frac{1}{[\varpi]}]$ = holomorphic functions on Y that are meromorphic along the divisors (π) and $([\varpi])$

$$= \Big\{ \sum_{n \gg -\infty} [x_n] \pi^n \;\Big/\; x_n \in F, \; \underbrace{\sup_n |x_n| < +\infty}_{\text{meromorphic at } ([\varpi])} \Big\}$$

$n \gg -\infty$ → meromorphic at (π)

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The fact is that $A[\frac{1}{\pi}, \frac{1}{[\varpi]}]$ is dense in $\mathcal{O}(Y)$ for the topology of "uniform convergence on quasi-compact open subsets"

<u>More precisely</u>:

<u>Gauss norms</u>: $\rho \in]0,1[$ "Radius"

Define for $f = \sum_{n \gg -\infty} [x_n] \pi^n \in A[\frac{1}{\pi}, \frac{1}{[\varpi]}]$

$$\|f\|_\rho = \sup_{n \in \mathbb{Z}} |x_n| \rho^n$$

<u>Def.</u> * $B := \mathcal{O}(Y) =$ Completion of $A[\frac{1}{\pi}, \frac{1}{[\varpi]}]$ w.r.t. to $(\|\cdot\|_\rho)_{\rho \in]0,1[}$

E-Frechet algebra

$\rightarrow$ a sequence is Cauchy if it is Cauchy $\forall \rho \in]0,1[$

Frechet completion

* $I \subset]0,1[$ interval

$$B_I = \text{Completion of } A\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right]$$
$$\text{w.r.t. } (|\cdot|_\rho)_{\rho \in I}$$

= "holo. fct. on the annulus defined by I"

$B = B_{]0,1[}$

* <u>Maximum modulus principle</u> (easy to check):

$$0 < \rho_1 \leq \rho \leq \rho_2 < 1$$
$$|f|_\rho \leq \sup\{|f|_{\rho_1}, |f|_{\rho_2}\}$$

$\Rightarrow$ If $I = [\rho_1, \rho_2] \subset]0,1[$ compact interval then

$$B_I = \text{Completion of } A\left[\frac{1}{\pi}, \frac{1}{[\varpi]}\right] \text{ w.r.t. } \sup\{|\cdot|_{\rho_1}, |\cdot|_{\rho_2}\}$$

= Banach E-algebra

$\rightarrow$ linked to the fact that Annulus defined by I is quasicompact

This is in fact a consequence of "Hadamard's 3 circles theorem".

Recall: $\mathcal{C} = \{ z \in \mathbb{C} / R < |z| < R' \}$

$f \in \mathcal{O}(\mathcal{C})$. Then the function

$$]R, R'[\longrightarrow \mathbb{R}_+$$
$$\rho \longmapsto \log \left(\sup_{|z|=\rho} |f(z)| \right)$$

is a convex function of $\log \rho$.

Here: non-archimedean analog

$\rho \in]0, 1[$ write $\rho = q^{-r}$ $r \in]0, +\infty[$

for $f \in A[\frac{1}{\pi}, \frac{1}{[\varpi]}]$ define $\boxed{v_r(f) = \inf_{n \in \mathbb{Z}} v(x_n) + nr}$

$\overset{\shortparallel}{\sum_n [x_n] \pi^n}$ then $|f|_\rho = q^{-v_r(f)}$

Easy: $]0, +\infty[\longrightarrow \mathbb{R}$ is concave

$r \longmapsto v_r(f)$

The case $E = \mathbb{F}_q((\pi))$:

In this case one checks that

$$Y = \mathbb{D}_F^* = \{0 < |\pi| < 1\} \subset \mathbb{A}_F^{1,\mathrm{ad}}$$ open punctured disk

$$\mathcal{O}(Y) = \left\{ \sum_{n\in\mathbb{Z}} x_n \pi^n \,/\, x_n \in F,\ \forall \rho \in]0,1[\ \lim_{|n|\to\infty} |x_n|\rho^n = 0 \right\}$$

$|\cdot|_\rho$ = supremum norm on $\{|\pi| = \rho\} \subset Y$

Rem: $\begin{matrix} Y \\ \downarrow \\ \mathrm{Spa}\, E \end{matrix}$. Here if $E = \mathbb{F}_q((\pi))$ we don't see ~~[illegible]~~ $\mathbb{D}_F^*$ as a classical F-rigid space but rather

$$\begin{matrix} \mathbb{D}_F^* \\ \downarrow \\ \mathrm{Spa}\, \mathbb{F}_q((\pi)) \end{matrix} \quad \text{via } \mathbb{F}_q((\pi)) \subset \mathcal{O}(Y)$$

$$\mathbb{D}_F^* = \mathrm{Spa}(F) \times_{\mathrm{Spa}(\mathbb{F}_q)} \mathrm{Spa}(E)$$

(categorical product in the category of $\mathbb{F}_q$-adic spaces)

$$\begin{tikzcd} & \mathbb{D}_F^* \arrow[dl] \arrow[dr] & \\ \mathrm{Spa}(F) & & \mathrm{Spa}(E) \end{tikzcd}$$

"classical" structural morphism locally of finite type

the structural morphism we are interested in: not locally of finite type.